Indian Statistical Institute, Bangalore Centre.
End-Semester Exam : Graph Theory

Instructor : Yogeshwaran D.

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Max. points : 40. **Time Limit : 3 hours.**

Answer any four questions only.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly.

Some notations :

- G is assumed to be a finite, undirected, simple graph everywhere.
- $P(G, x)$ - chromatic polynomial of the graph G.
- I - Identity matrix. J - Matrix with all entries 1. L - Laplacian matrix of the graph.
- $A(G)$ - Adjacency matrix of a graph G.

1. Show that the number of spanning trees of G is $n^{-2} det(L + J)$. (**10**)
2. Let $p_r(G)$ denote the number of partitions of $V(G)$ into r non-empty independent sets. Show that $P(G, x) = \sum_{r=1}^{n} p_r(G)x_{(r)}$ where $x_{(r)} = x(x-1)\dots(x-r+1)$ and $n = |V(G)|$. (**10**)
3. Let T be a tree. Show that determinant of every square submatrix of $A(T)$ equals one of $0, -1, +1$. Also, when is $det\, A(T) \neq 0$?(**10**)
4. Let G be a finite undirected graph and u and v two distinct vertices. Then the size of the minimum (u, v)-edge cut is equal to the maximum number of pairwise edge-disjoint paths from u to v. (**10**)
(**Note :** You may assume the appropriate max-flow min-cut theorem but state it explicitly.)
5. (a) Show that every 3-regular graph with no cut-edge has a 1-factor. (**5**)

(b) Show that a tree has at most one perfect matching. (**5**)

6. (a) A plane graph is a k-angulation if every face has length k. Let G be a connected k-angulation on n vertices and m edges. Show that $m = (n-2)\frac{k}{k-2}$. (**4**)

   (b) Is there a K_5 minor of the Petersen graph ? (**2**)

   (c) Show that the Petersen graph has a $K_{3,3}$ minor. (Hint : Start by deleting one vertex and do not delete any edges.) (**4**).